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ENCOUNTER-EVASION PROBLEMS IN QUASIDYNAMIC SYSTEMS

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Encounter-evasion game problems in quasidynamic and semidynamic systems are analyzed. Theorems on the alternative in the class of piecewise-program strategies of the players are stated and proved. The work adjoins the researches in [1 — 9].

1. Let certain nonempty sets X, U and V exist. Set X is called the state set; $U(V)$ is the first (second) player's set of instantaneous values of the controls. Let $D_1 (D_2)$ be some nonempty set of mappings on the half-open interval $[t_0, T)$ into $U(V)$ and let some mapping $\varkappa$ of set $[t_0, T) \times D_1 \times D_2$ into X be given. Set $D_1 (D_2)$ is called the set of admissible controls of the first (second) player and the mapping $\varkappa$ is called the state function. The quintuple $\Sigma = ([t_0, T), X, D_1, D_2, \varkappa)$ is called a quasidynamic system if the following condition is fulfilled:

Condition 1). For any admissible controls $u_1, u_2 \in D_1$ and $v_1, v_2 \in D_2$ of the players and for any instants $t_0 \leqslant t_1 < t_2 < t \leqslant T$ there exist admissible controls $u_3 \in D_1$ and $v_3 \in D_2$ such that

$$u_3(t) = \begin{cases} u_1(t), & t_1 \leqslant t < t_2, \\ u_2(t), & t_2 \leqslant t < t_3, \end{cases} \quad v_3(t) = \begin{cases} v_1(t), & t_1 \leqslant t < t_2 \\ v_2(t), & t_2 \leqslant t < t_3 \end{cases}$$

An element $x(t) = \varkappa(t, u, v)$ of set X is called a state of system Σ at instant t and the mapping $x(\cdot) = \varkappa(\cdot, u, v)$ of the half-open interval $[t_0, T)$ into set X is called the trajectory of this system, corresponding to the pair of controls u and v. The concept of the players' piecewise-program strategies can be introduced for quasidynamic systems analogously as in the theory of differential games [6]. By $D_1[t_1, t_2)$ $(D_2[t_1, t_2))$ we denote the set of all restrictions of the first (second) player's admissible controls to the half-open interval $[t_1, t_2) \subset [t_0, T)$. Let $\Delta = \{t_0 = t_0^\Delta < t_1^\Delta < \ldots < t_{n(\Delta)}^\Delta = T\}$ be an arbitrary finite partitioning of the half-open interval $[t_0, T)$. The set of all finite partitionings of the half-open interval $[t_0, T)$ is denoted $\{\Delta\}$.

The sequence $\varphi_\Delta = (\varphi_{\Delta,1}, \ldots, \varphi_{\Delta,n(\Delta)})$, where $\varphi_{\Delta,1} \in D_1[t_0, t_1^\Delta)$ and $\varphi_{\Delta,k}$ $(k \geqslant 2)$ is any mapping of set $D_1[t_0, t_{k-1}^\Delta) \times D_2[t_0, t_{k-1}^\Delta)$ into $D_1[t_{k-1}^\Delta, t_k^\Delta)$, is called the first player's Δ-strategy. The pair $\varphi = (\Delta, \varphi_\Delta)$, where $\Delta \in \{\Delta\}$ and φ_Δ is any Δ-strategy of the first player, is called the first player's piecewise-program strategy. The sequence $\varphi^\Delta = (\varphi^{\Delta,1}, \ldots, \varphi^{\Delta,n(\Delta)})$, where $\varphi^{\Delta,k}$ is any mapping of set $D_1[t_0, t_{k-1}^\Delta) \times D_2[t_0, t_k^\Delta)$ into $D_1[t_{k-1}^\Delta, t_k^\Delta)$ $(k = 1, 2, \ldots, n(\Delta))$ is called the first player's upper Δ-strategy. The Δ-strategy ψ_Δ, the piecewise-program strategy $\psi = (\Delta, \psi_\Delta)$ and the upper Δ-strategy ψ^Δ of the second player are defined similarly. By $D_{1\Delta}$ $(D_{2\Delta})$ we denote the set of all Δ-strategies, by $D_1^* (D_2^*)$ the set of all piecewise-program strategies and by $D_1^\Delta(D_2^\Delta)$, the set of all upper Δ-strategies of the first (second) player.

As for differential games [9], any pair of strategies φ^Δ and ψ_Δ defines a unique pair of controls

$$u^\Delta = u(\varphi^\Delta, \psi_\Delta) \in D_1, \; v_\Delta = v(\varphi^\Delta, \psi_\Delta) \in D_2$$

and, consequently, determines a unique trajectory

$$x(t) = \varkappa(t, \varphi^\Delta, \psi_\Delta) = \varkappa(t, u^\Delta, v_\Delta)$$

of system Σ. Analogously, any pair of strategies φ_Δ and ψ^Δ defines a unique trajectory

$$x(t) = \varkappa(t, \varphi_\Delta, \psi^\Delta) = \varkappa(t, u(\varphi_\Delta, \psi^\Delta), v(\varphi_\Delta, \psi^\Delta))$$

and any pair of strategies φ and ψ defines a unique trajectory

$$x(t) = \varkappa(t, \varphi, \psi) = \varkappa(t, u(\varphi, \psi), v(\varphi, \psi))$$

of system Σ.

Let $\Phi(\Sigma)$ be the set of all trajectories of the quasidynamic system Σ and let a certain functional g be given on the set $\Phi(\Sigma) \times D_1 \times D_2$. Then, the functional

$$I = I(u, v) = g(\varkappa(\cdot, u, v), u, v) \tag{1.1}$$

is defined on the set $D_1 \times D_2$. This functional is called the first player's payoff; the functional $-I$ is called the second player's payoff. The mapping (1.1) defines the functionals

$$I = I(\varphi^\Delta, \psi_\Delta) = I(u(\varphi^\Delta, \psi_\Delta), v(\varphi^\Delta, \psi_\Delta)) \tag{1.2}$$

on the sets $D_1^\Delta \times D_{2\Delta}$

$$I = I(\varphi_\Delta, \psi^\Delta) = I(u(\varphi_\Delta, \psi^\Delta), v(\varphi_\Delta, \psi^\Delta)) \tag{1.3}$$

on the set $D_{1\Delta} \times D_2^\Delta$ and

$$I = I(\varphi, \psi) = I(u(\varphi, \psi), v(\varphi, \psi)) \tag{1.4}$$

on the set $D_1^* \times D_2^*$.

Definition 1.1. The triple $\Gamma = \langle I, D_1^*, D_2^* \rangle$ is called an antagonistic quasidynamic game. The quantity

$$\mathrm{V}^* = \inf_{\psi \in D_2^*} \sup_{\varphi \in D_1^*} I(\varphi, \psi)$$

is called the upper value and the quantity

$$\mathrm{V}_* = \sup_{\varphi \in D_1^*} \inf_{\psi \in D_2^*} I(\varphi, \psi)$$

is called the lower value of the game Γ. We say that game Γ has a value if the equality

$$\mathbf{V}^* = \mathbf{V}_* = \text{val}\, \Gamma$$

is valid.

The triple $\Gamma^\Delta = \langle I, D_1^\Delta, D_{2\Delta}\rangle$ $(\Gamma_\Delta = \langle I, D_{1\Delta}, D_2^\Delta\rangle)$ is called an upper (a lower) Δ-game. In these games one of the players is discriminated against. We introduce the notation

$$\mathbf{V}^\Delta = \inf_{\psi_\Delta \in D_{2\Delta}} \sup_{\varphi^\Delta \in D_1^\Delta} I(\varphi^\Delta, \psi_\Delta)$$
$$\mathbf{V}_\Delta = \sup_{\varphi_\Delta \in D_{1\Delta}} \inf_{\psi^\Delta \in D_2^\Delta} I(\varphi_\Delta, \psi^\Delta)$$

The following statement is valid.

Lemma 1.1. If $\Delta_1 \subset \Delta_2$, then

$$\mathbf{V}^{\Delta_1} \geqslant \mathbf{V}^{\Delta_2} \geqslant \mathbf{V}^* \geqslant \mathbf{V}_* \geqslant \mathbf{V}_{\Delta_2} \geqslant \mathbf{V}_{\Delta_1}$$

From this lemma it follows that the limits

$$\mathbf{V}_+ = \lim_{n\to\infty} \mathbf{V}^{\omega(n)}, \quad \mathbf{V}_- = \lim_{n\to\infty} \mathbf{V}_{\omega(n)}$$

exist, where $\{\omega(n)\}$, $n = 1, 2, \ldots$, is a sequence of partitionings of the form

$$\omega(n) = \{t_k^n \mid t_k^n = t_0 + k\delta(n),\ k = 0, 1, \ldots, 2^n\}, \quad \delta(n) = \frac{T - t_0}{2^n}$$

and if $\mathbf{V}_+ = \mathbf{V}_-$, the quasidynamic game Γ has the value

$$\text{val}\, \Gamma = \mathbf{V}_+ = \mathbf{V}_-$$

As in [9] it can be shown that all upper and lower Δ-games have the values $\mathbf{V}^\Delta = \text{val}\, \Gamma^\Delta$ and $\mathbf{V}_\Delta = \text{val}\, \Gamma_\Delta$.

2. Let us consider encounter-evasion games [1 – 3]. Let the state set X of system Σ be a metric space with metric d. For any set $K \subset [t_0, T] \times X$ we denote its ε-neighborhood in $[t_0, T) \times X$ by K^ε. We formulate the following two problems.

Encounter Problem 2.1. For any number $\varepsilon > 0$ find the first player's piecewise-program strategy φ_ε such that the relations

$$\{\tau, x(\tau)\} \in M^\varepsilon,\ \{t, x(t)\} \in N^\varepsilon,\ t_0 \leqslant t < \tau = \tau(\varphi_\varepsilon, \psi) \leqslant T \qquad (2.1)$$

are fulfilled for all trajectories

$$x(t) = \varkappa(t, \varphi_\varepsilon, \psi),\ \psi \in D_2^*$$

Evasion Problem 2.2. Find a number $\varepsilon > 0$ and a second player's piecewise-program strategy ψ_ε excluding the contact (2.1) for any trajectory

$$x(t) = \varkappa(t, \varphi, \psi_\varepsilon),\ \varphi \in D_1^*$$

On the set of trajectories of the quasidynamic system Σ we intruduce the uniform metric

$$\rho[x_1(\cdot), x_2(\cdot)] = \sup_{T_0 \leqslant t < T} d[x_1(t), x_2(t)] \tag{2.2}$$

We state the following conditions.

B. 1. Let $\{u^{\delta(n)}\}$ be any sequence of admissible controls of the first player, $\delta(n) = (T - t_0)/2^n$, $\{n\} \subset \{1, 2, \ldots\}$; let $u_*^{\delta(n)}(t) = u^{\delta(n)}(t - \delta(n))$ for $t_0 + \delta(n) \leqslant t < T$ and $u_*^{\delta(n)}(t)$ for $t_0 \leqslant t < t_0 + \delta(n)$ be restrictions of admissible controls. Then a number $\eta > 0$ exists such that $u_*^{\delta(n)} \in D_1$ if only $\delta(n) < \eta$;

B. 2. $\rho[\varkappa(\cdot, u^{\delta(n)}, v^{\delta(n)}), \varkappa(\cdot, u_*^{\delta(n)}, v^{\delta(n)})] \to 0$ as $n \to \infty$ uniformly relatively to all $u^{\delta(n)}, u_*^{\delta(n)} \in D_1$ and $v^{\delta(n)} \in D_2$.

The following statement is valid.

Theorem 2.1. If a quasidynamic system Σ satisfies Conditions B.1 and B.2, then either the Encounter Problem 2.1 or the Evasion Problem 2.2 is solvable for it.

Proof. Let us consider the family of upper $\omega(n)$-games $\Gamma_\varepsilon^{\omega(n)}$ in which the first player's payoff has the form

$$I_\varepsilon = - \inf_{t_0 \leqslant t < \tau_\varepsilon^N(u, v)} \operatorname{dist}[\{t, x(t)\}, M], \qquad \varepsilon > 0 \tag{2.3}$$

$$\tau_\varepsilon^N = \inf\{t_0 \leqslant t < T \mid \{t, x(t)\} \in [([t_0, T) \times X) \setminus N^\varepsilon]\}$$

$$\operatorname{dist}[\{t, x\}, M] = \inf_{\{t_*, x_*\} \in M} \{|t - t_*| + d[x, x_*]\}$$

$$x(t) = \varkappa(t, u, v)$$

Let a number $\varepsilon > 0$ exists such that

$$\mathrm{V}_+(\varepsilon) = \lim_{n\to\infty} \mathrm{V}_\varepsilon^{\omega(n)} = \lim_{n\to\infty} \operatorname{val} \Gamma_\varepsilon^{\omega(n)} < 0$$

Then the second player's $\omega(n)$-strategies $\psi^\varepsilon_{\omega(n)}$ and a number $N_1(\varepsilon)$ exist satisfying the condition

$$I(\varphi, \psi^\varepsilon_{\omega(n)}) < \mathrm{V}_+(\varepsilon)/2, \; n > N_1(\varepsilon)$$

for all $\varphi \in D_1^*$. Consequently

$$\inf_{t_0 \leqslant t < \tau_\varepsilon^N(\varphi, \psi^\varepsilon_{\omega(n)})} \operatorname{dist}[\{t, \varkappa(t, \varphi, \psi^\varepsilon_{\omega(n)})\}, M] > -\frac{\mathrm{V}_+(\varepsilon)}{2} \tag{2.4}$$

for all $\varphi \in D_1^*$ when $n > N_1(\varepsilon)$. If $-\mathrm{V}_+(\varepsilon)/2 \geqslant \varepsilon$, then from inequalities (2.4) it follows that the strategies $\psi^\varepsilon_{\omega(n)}$ $(n > N_1(\varepsilon))$ solve Evasion Problem 2.2. It is easy to show that these strategies solve Evasion Problem 2.2 also when $-\mathrm{V}_+(\varepsilon)/2 < \varepsilon$.

It remains to consider the case when the relation

$$\mathrm{V}_+(\varepsilon) = 0 \leqslant \operatorname{val} \Gamma_\varepsilon^{\omega(n)} \leqslant 0, \quad n = 1, 2, \ldots$$

is fulfilled for all numbers $\varepsilon > 0$. In this case there exist the first player 's upper $\omega(n)$-strategies $\varphi_\varepsilon^{\omega(n)}$ satisfying the inequality

$$-\varepsilon/2 < I_{\varepsilon/2}(\varphi_\varepsilon^{\omega(n)}, \psi) \leqslant 0$$

for all $\psi \in D_2^*$. Consequently

$$\inf_{t_0 \leqslant t < \tau^N_{\varepsilon/2}(\varphi_\varepsilon^{\omega(n)}, \psi)} \operatorname{dist}[\{t, \varkappa(t, \varphi_\varepsilon^{\omega(n)}, \psi)\}, M] < \frac{\varepsilon}{2} \tag{2.5}$$

for all $\psi \in D_2^*$. From Condition B.1 it follows that for the strategies $\varphi_\varepsilon^{\omega(n)}$ and for any upper $\omega(n)$ -strategies of the second player we can construct $\omega(n)$ strategies $\varphi_\varepsilon(t - \delta(n))$ and $\psi^*_{\omega(n)}$ such that if

$$u_1 = u(\varphi_\varepsilon(t - \delta(n)), \psi^{\omega(n)}), \; u_2 = u(\varphi_\varepsilon^{\omega(n)}, \psi^*_{\omega(n)})$$

then

$$u_1(t) = u_2(t - \delta(n)), \; t_0 + \delta(n) \leqslant t < T$$

$$v(\varphi_\varepsilon(t - \delta(n)), \psi^{\omega(n)}) = v(\varphi_\varepsilon^{\omega(n)}, \psi^*_{\omega(n)})$$

A method for constructing such strategies has been described in [9]. By Condition B. 2 we can choose a number $N_2(\varepsilon)$ such that

$$\rho[\varkappa(\cdot, \varphi_\varepsilon^{\omega(n)}, \psi^*_{\omega(n)}), \varkappa(\cdot, \varphi_\varepsilon(t - \delta(n)), \psi^{\omega(n)})] < \frac{\varepsilon}{2} \tag{2.6}$$

for all $\psi^{\omega(n)} \in D_2^{\omega(n)}$ when $n > N_2(\varepsilon)$. From inequalities (2.6) and (2.5) we get that the strategies $\varphi_\varepsilon(t - \delta(n))$, $n > N_2(\varepsilon)$, solve Encounter Problem 2.1. Thus, we have proved a statement even somewhat stronger than Theorem 2.1 since for all $n = 1, 2, \ldots$

$$D_{1\omega(n)} \subset D_1^* \subset D_1^{\omega(n)}, \; D_{2\omega(n)} \subset D_2^* \subset D_2^{\omega(n)}$$

3. The quintuple $\Sigma = ([t_0, T], X, D_1, D_2, \varkappa)$, where $\varkappa$ is a mapping of set $[t_0, T] \times [t_0, T] \times X \times D_1 \times D_2$ into X and D_1 and D_2 satisfy Condition 1), is called a dynamic system in the sense of Kalman if it satisfies the conditions:

2) if $u_1, u_2 \in D_1$, $v_1, v_2 \in D_2$, $u_1(s) = u_2(s)$ and $v_1(s) = v_2(s)$ for $t_0 \leqslant t_1 \leqslant s < t_2 \leqslant T$, then for any $x \in X$

$$\varkappa(t_2, t_1, x, u_1, v_1) = \varkappa(t_2, t_1, x, u_2, v_2)$$

3) $\varkappa(t, t, x, u, v) = x$ for all $t_0 \leqslant t \leqslant T$, $x \in X$, $u \in D_1$ and $v \in D_2$;

4) the relation

$$\varkappa(t_3, t_1, x, u, v) = \varkappa(t_3, t_2, \varkappa(t_2, t_1, x, u, v), u, v)$$

is valid for any $t_0 \leqslant t_1 < t_2 < t_3 \leqslant T$ and for any $x \in X$, $u \in D_1$ and $v \in D_2$;

5) the mapping $x = \varkappa(t, \tau, x_*, u, v)$ is defined for all $t \geqslant \tau$ and is not necessarily defined for $t < \tau$.

The element $x(t) = \varkappa(t, t_*, x_*, u, v)$ of set X is called a state of system Σ at instant t and the corresponding mapping $x(\cdot)$: $[t_0, T] \to X$ is called a trajectory of system Σ if this system is found in state x_* at instant t_* and controls u and v act on it.

Any dynamic system $\Sigma = ([t_0, T], X, D_1, D_2, \varkappa)$ defines the quasidynamic system

$$\Sigma(t_*, x_*) = ([t_*, T], X, D_1[t_*, T), D_2[t_*, T), \varkappa_*)$$

with state function $\varkappa_*(t, u, v) = \varkappa(t, t_*, x_*, u, v)$, for each fixed initial state $x(t_*) = x_*$. The set $[t_0, T] \times X$ is called the position set. For each fixed position $\{t_*, x_*\}$ let the functional

$$I = g(x(\cdot), u, v, t_*, x_*)$$

be given on the set $\Phi(\Sigma(t_*, x_*)) \times D_1 \times D_2$, where $\Phi(\Sigma(t_*, x_*))$ is the set of all trajectories of system $\Sigma(t_*, x_*)$. Then the functional

$$I = I(u, v, t_*, x_*) = g(\varkappa(\cdot, t_*, x_*, u, v), u, v, t_*, x_*) \tag{3.1}$$

has been defined on $D_1 \times D_2 \times [t_0, T] \times X$, which we call the first player's payoff at position $\{t_*, x_*\}$.

Definition 3.1. A quasidynamic game described by system $\Sigma(t_*, x_*)$, in which the first player's payoff has the form (3.1), is called a dynamic (k -dynamic) game

$$\Gamma(t_*, x_*) = \langle I, D_1^*[t_*, T), D_2^*[t_*, T)\rangle$$

described by system Σ, in which the first player has the same payoff.

The corresponding upper (lower) Δ-games

$$\Delta = \{t_* = t_0^\Delta < t_1^\Delta < \ldots < t_{n(\Delta)}^\Delta = T\}$$
$$V^\Delta(t_*, x_*) = \mathrm{val}\,\Gamma^\Delta(t_*, x_*), \quad V_\Delta(t_*, x_*) = \mathrm{val}\,\Gamma_\Delta(t_*, x_*)$$
$$V(t_*, x_*) = \mathrm{val}\,\Gamma(t_*, x_*)$$

are denoted by the symbol $\Gamma^\Delta(t_*, x_*)\,(\Gamma_\Delta(t_*, x_*))$.

4. We introduce the following concepts.

Definition 4.1. The vector $\varphi_\Delta = (\varphi_{\Delta,1}, \ldots, \varphi_{\Delta,n(\Delta)})$, where $\varphi_{\Delta,1} \in D_1[t_*, t_1^\Delta)$ and $\varphi_{\Delta,k}\colon X \to D_1[t_{k-1}^\Delta, t_k^\Delta)$, $k = 2, 3, \ldots, n(\Delta)$, is called a position Δ-strategy and the vector $\varphi^\Delta = (\varphi^{\Delta,1}, \ldots, \varphi^{\Delta,n(\Delta)})$, where

$$\varphi^{\Delta,k}\colon X \times D_2[t_{k-1}^\Delta, t_k^\Delta) \to D_1[t_{k-1}^\Delta, t_k^\Delta), \; k = 1, 2, \ldots, n(\Delta),$$

is called a position upper Δ-strategy of the first player in system $\Sigma(t_*, x_*)$. The pair $\varphi = (\Delta, \varphi_\Delta)$, where Δ is an arbitrary finite partitioning of the interval $[t_*, T]$ and φ_Δ is any position Δ-strategy of the first player in system $\Sigma(t_*, x_*)$, is called a position piecewise-program strategy of the first player in system $\Sigma(t_*, x_*)$.

The position Δ-strategies ψ_Δ, the position upper Δ-strategies ψ^Δ and the position piecewise-program strategies $\psi = (\Delta, \psi_\Delta)$ of the second player in system $\Sigma(t_*, x_*)$ are defined similarly.

Piecewise-program strategies were first introduced in differential game theory in precisely such a form. We note that the classes $D_1^*[t_*, T)$ and $D_2^*[t_*, T)$ of the player's piecewise-program strategies, examined in the present paper, contain a wider class of strategies.

Let the state set of dynamic system Σ be a metric space with metric d. For any set $K \subset [t_0, T] \times X$ we denote its ε-neighborhood in $[t_0, T]$ by K^ε. Let there be certain sets M and N in $[t_0, T] \times X$ and let the game's initial position $\{t_*, x_*\}$ be given. We examine the following two problems.

Encounter Problem 4.1. For any number $\varepsilon > 0$ find the first player's position piecewise-program strategy φ_ε such that the relations

$$\{\tau, x(\tau)\} \in M^\varepsilon, \ \{t, x(t)\} \in N^\varepsilon \qquad (4.1)$$
$$t_* \leqslant t < \tau = \tau[x(\cdot)] \leqslant T$$

are fulfilled for all trajectories

$$x(t) = \varkappa(t, t_*, x_*, \varphi_\varepsilon, \psi), \quad \psi \in D_2^*[t_*, T)$$

Evasion Problem 4.2. Find a number $\varepsilon > 0$ and a second player's position piecewise-program strategy ψ_ε such that contact (4.1) is excluded for all trajectories

$$x(t) = \varkappa(t, t_*, x_*, \varphi, \psi_\varepsilon), \quad \varphi \in D_1^*[t_*, T)$$

We consider dynamic systems Σ satisfying the condition

C. 1. For any $t_0 \leqslant t_1 < t_2 \leqslant T$ and $x_1, x_2 \in X$ the controls $u_* = u(t_1, t_2, x_1, x_2) \in D_1[t_1, t_2)$ and $v_* = v(t_1, t_2, x_1, x_2) \in D_2[t_1, t_2)$ exist such that

$$d^m[\varkappa(t, t_1, x_1, u_*, v), \ \varkappa(t, t_1, x_2, u, v_*)] \leqslant d^m[x_1, x_2] \times \qquad (4.2)$$
$$\exp\beta(t - t_1) + \gamma(t - t_1)(t - t_1) \quad \lim_{\delta\to 0}\gamma(\delta) = 0, \quad m, \beta > 0$$
$$t_1 \leqslant t \leqslant t_2$$

for all $u \in D_1$ and $v \in D_2$, where d is some metric on the state set X.

The following statement is valid.

Theorem 4.1. If a dynamic system Σ satisfies condition C.1, then either the Encounter Problem 4.1 or the Evasion Problem 4.2 is solvable for any position $\{t_*, x_*\}$ of this system.

Proof. Theorem 4.1 is proved similarly to Theorem 2.1. The position character of the piecewise-program strategies φ_ε and ψ_ε follows from Condition 1) – 5) and from the form of the payoff functional

$$I = -\inf_{t_* \leqslant t < \tau_\varepsilon^N(u, v, t_*, x_*)} \operatorname{dist}[\{t, x(t)\}, M]$$

in the auxiliary games $\Gamma_\varepsilon^{\omega(n)}(t_*, x_*)$.

5. Let us consider dynamic systems satisfying the conditions

6) $U \subset D_1, \ V \subset D_2$

C. 2. For all $t_0 \leqslant t_1 < T$ and $x_1, x_2 \in X$ the controls $u_* = u(t_1, x_1, x_2) \in U$ and $v_* = v(t_1, x_1, x_2) \in V$ exist such that condition (4.2) is fulfilled for all $u \in D_1$ and $v \in D_2$.

Definition 5.1. A piecewise-program strategy $\varphi = (\Delta; \varphi_{\Delta,1}, \ldots, \varphi_{\Delta,n(\Delta)})$

$(\psi = (\Delta; \psi_{\Delta,1}, \ldots, \psi_{\Delta,n(\Delta)}))$ of the first (second) player in system $\Sigma\ (t_*, x_*)$ is said to be piecewise-constant if the mappings $\varphi_{\Delta,k}\ (\psi_{\Delta,k})$, $k = 1, 2, \ldots, n(\Delta)$, take values in set $U\ (V)$.

The first (second) player's position piecewise-constant strategy

$$\varphi = (\Delta, \varphi_\Delta) \quad (\psi = (\Delta, \psi_\Delta))$$

in game $\Sigma\ (t_*, x_*)$ can be identified with the mapping $u_\Delta(t, x)$ $(v_\Delta(t, x))$ of set $[t_0, T] \times X$ into $U\ (V)$ such that

$$\varphi_{\Delta,k+1} = u\ (t_k{}^\Delta, x) \quad (\psi_{\Delta,k+1} = v\ (t_k{}^\Delta, x))$$
$$k = 0, 1, \ldots, n(\Delta) - 1$$

We state two problems.

Encounter Problem 5.1. For any number $\varepsilon > 0$ find the first player's position piecewise-constant strategy such that relations (4.1) are fulfilled for all trajectories

$$x\ (t) = \varkappa\ (t, t_*, x_*, u_\Delta{}^\varepsilon\ (t, x), \psi), \quad \psi \in D_2{}^*\ [t_*, T)$$

Evasion Problem 5.2. Find a number $\varepsilon > 0$ and a second player's position piecewise-constant strategy $v_\Delta{}^\varepsilon\ (t, x)$ excluding contact (4.1) for all trajectories

$$x\ (t) = \varkappa\ (t, t_*, x_*, \varphi, v_\Delta{}^\varepsilon\ (t, x)), \quad \varphi \in D_1{}^*\ [t_*, T)$$

The proof of the next statement is similar to that of Theorem 4.1.

Theorem 5.1. If a dynamic system Σ satisfies Conditions 6) and C.2, then either the Encounter Problem 5.1 or the Evasion Problem 5.2 is solvable for any position $\{t_*, x_*\}$ of this system.

6. We introduce the following concept.

Definition 6.1. Any mapping

$$u\ (t, x): [t_0, T] \times X \to U \quad (v\ (t, x) : [t_0, T] \times X \to V)$$

is called a position strategy of the first (second) players in system Σ.

For any position $\{t_*, x_*\}$ of system Σ and for any finite partitioning Δ of the interval $[t_*, T]$ the pair $\{\Delta, u\ (t, x)\}$ $(\{\Delta, v\ (t, x)\})$, where $u\ (t, x)$ $(v\ (t, x))$ is a position strategy, can be treated as a position piecewise-constant strategy of the first (second) player in system $\Sigma\ (t_*, x_*)$.

We examine the following two problems.

Encounter Problem 6.1. Find the first player's position strategy $u\ (t, x)$ possessing the property: for any number $\varepsilon > 0$ a number $\delta > 0$ exists such that relations (4.1) are fulfilled for all trajectories

$$x\ (t) = \varkappa\ (t, t_*, x_*, \{\Delta, u\ (t, x)\}, \psi), \quad \psi \in D_2{}^*\ [t_*, T)$$
$$|\Delta| = \max_{k=0,1,\ldots,n(\Delta)-1} (t_{k+1}^\Delta - t_k{}^\Delta) < \delta$$

Evasion Problem 6.2. Find numbers $\varepsilon > 0$ and $\delta > 0$ and a second player's position strategy $v(t, x)$ excluding contact (4.1) for all trajectories

$$x(t) = \varkappa(t, t_*, x_*, \varphi, \{\Delta, v(t, x)\}),\ \varphi \in D_2^*[t_*, T),\ |\Delta| < \delta .$$

Let us consider dynamic systems Σ satisfying the conditions

7) the state set X is a compact metric space with metric d;

8) for all $\{t_1, x_1\} \in [t_0, T] \times X$ and for any number $\varepsilon > 0$ a number $\delta = \delta(t_1, x_1, \varepsilon)$ exists such that

$$\sup_{(u,v,t)\in D_1\times D_2\times[t_*,T]} d[\varkappa(t, t_1, x_1, u, v), \varkappa(t, t_1, x_2, u, v)] \leqslant \varepsilon$$

if only $d[x_1, x_2] \leqslant \delta$.

The following statement is valid.

Theorem 6.1. If a dynamic system Σ satisfies Conditions 1) – 8) and C.2, then either the Encounter Problem 6.1 or the Evasion Problem 6.2 is solvable for any position $\{t_*, x_*\}$ of this system.

To prove this theorem we use stable bridges similar to those in the theory of position differential games [1 – 3].

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